Newton’s Method

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Define,

$$N(f, x_0) = \begin{cases} x_0 - f(x_0)/f'(x_0) & \text{if } f'(x_0) \neq 0, \\ x_0 & \text{if } f'(x_0) = 0. \end{cases}$$

Convergence

$$\lim_{k \to \infty} N_k(f, x_0) = \xi$$

a root of $f$. 
Newton’s Method

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Define,

$$N(f, x_0) = \begin{cases} x_0 - f(x_0)/f'(x_0) & \text{if } f'(x_0) \neq 0, \\ x_0 & \text{if } f'(x_0) = 0. \end{cases}$$

$$N^k(f, x_0) = (N \circ N \circ \cdots \circ N)(f, x_0)$$

$k$ times
Newton’s Method

Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial. Define,

$$N(f, x_0) = \begin{cases} 
  x_0 - f(x_0)/f'(x_0) & \text{if } f'(x_0) \neq 0, \\
  x_0 & \text{if } f'(x_0) = 0. 
\end{cases}$$

$$N^k(f, x_0) = (N \circ N \circ \cdots \circ N)(f, x_0)$$

Convergence

$$\lim_{k \to \infty} N^k(f, x_0) = \xi \text{ a root of } f$$
Roots of a Simple Cubic

$$f(x) = x^3 - 1$$

Actual roots:

$$\xi_k = e^{2\pi ik/3}, \text{ for } k = 0, 1, 2.$$
Example: Actual Roots
Example: What is a “Good Guess”?

An initial guess “close” to the root should converge to that root:
Example: What is a “Good Guess”? 

An initial guess “close” to the root should converge to that root:

- white region → guesses converging to $\xi_0$,
- grey region → guesses converging to $\xi_1$,
- black region → guesses converging to $\xi_2$,

(Apply Newton’s Method to each guess until we reach a root.)
Example: What is a “Good Guess”? 

![Complex Plane Diagram]

- **Re(x)** represents the real part of the complex number.
- **Im(x)** represents the imaginary part of the complex number.

The diagram illustrates different regions in the complex plane, with points marked indicating various real and imaginary values.
Example: What is a “Bad Guess”? 

What if the initial Newton guess is further away?
Example: What is a “Bad Guess”? 

What if the initial Newton guess is further away?

Next Slide

- white region $\rightarrow$ guesses converging to $\xi_0$,
- grey region $\rightarrow$ guesses converging to $\xi_1$,
- black region $\rightarrow$ guesses converging to $\xi_2$, 
Example: What is a “Bad Guess”?
There are many terrible guesses.
Moral of Story

There are many terrible guesses.

(Even guesses closer to some roots converge to other roots.)
Example: Roots of $f(x) = x^9 - 1$
Two Questions

Question #1
Can we ensure our guesses are far away from nasty fractal areas?
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Can we ensure our guesses are far away from nasty fractal areas?

Question #2
Given two guesses can we determine if they will converge to different roots? (Or the same root?)
Two Questions

Question #1
Can we ensure our guesses are far away from nasty fractal areas?

Question #2
Given two guesses can we determine if they will converge to different roots? (Or the same root?)

But...
...can we do these a priori? (w/o knowing location of roots)
Terminology

Let \( f : \mathbb{C} \rightarrow \mathbb{C} \) be a polynomial.

- Define: \( x \in \mathbb{C} \) is an **approximate solution** to \( f \) with **associated solution** \( \xi \in \mathbb{C} \) if

\[
|N^{(k)}(f, x) - \xi| \leq \left( \frac{1}{2} \right)^{2^k-1} |x - \xi|
\]
Terminology

Let $f : \mathbb{C} \rightarrow \mathbb{C}$ be a polynomial.

- Define: $x \in \mathbb{C}$ is an **approximate solution** to $f$ with **associated solution** $\xi \in \mathbb{C}$ if

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- approximate solutions converge quadratically to their associated solutions
Let $f : \mathbb{C} \to \mathbb{C}$ be a polynomial.

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- approximate solutions converge quadratically to their associated solutions

- "$x$ lies inside the quadratic convergence region of $\xi$"
Figure: Quadratic convergence region of $\xi = 1$ for $f(x) = x^3 - 1$. 
Question #1: Ensuring Quadratic Convergence

Determine if $x$ is an approximate solution.
Determine if $x$ is an **approximate solution**.

**Problem:** the condition

$$
\left| N^{(k)}(f, x) - \xi \right| \leq \left( \frac{1}{2} \right)^{2^k - 1} |x - \xi|
$$

requires knowing $\xi$!
Determine if $x$ is an **approximate solution**.

- **Problem**: the condition

$$\left| N^{(k)}(f, x) - \xi \right| \leq \left( \frac{1}{2} \right)^{2^k - 1} |x - \xi|$$

requires knowing $\xi$!

- **Smale’s Alpha Theory**: sufficient conditions for $x$ to be in *some* quadratic convergence region
Smale’s alpha theory: let
Smale’s alpha theory: let

$$\alpha(f, x) := \beta(f, x)\gamma(f, x)$$
Smale’s alpha theory: let

\[ \alpha(f, x) := \beta(f, x) \gamma(f, x) \]
\[ \beta(f, x) := |x - N(f, x)| = |f(x)/f'(x)| \]
Smale’s alpha theory: let

\[ \alpha(f, x) := \beta(f, x) \gamma(f, x) \]
\[ \beta(f, x) := |x - N(f, x)| = \left| \frac{f(x)}{f'(x)} \right| \]
\[ \gamma(f, x) := \max_{k \geq 2} \left| \frac{f^{(k)}(x)/f'(x)}{k!} \right|^\frac{1}{k-1} \]
Smale Theorem #1

If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial and $x \in \mathbb{C}$ such that

$$\alpha(f, x) \leq \frac{13 - 3\sqrt{17}}{4} \approx 0.157671$$

then $x$ is an approximate solution to $f$. 
Smale Theorem #1
If \( f : \mathbb{C} \to \mathbb{C} \) is a polynomial and \( x \in \mathbb{C} \) such that
\[
\alpha(f, x) \leq \frac{13 - 3\sqrt{17}}{4} \approx 0.157671
\]
then \( x \) is an approximate solution to \( f \).
Additionally,
\[
|x - \xi| \leq 2\beta(f, x)
\]
where \( \xi \) is the associated solution to \( x \).
Figure: Region where $\alpha(f, x) < 0.157 \ldots$ for $f(x) = x^3 - 1$. 
Pros

- Quadratic convergence condition
- Doesn’t say which root (but \( \beta \) gives us an idea)
- Alpha region much smaller than quadratic convergence region
Pros
- quadratic convergence condition *without* knowing roots,
Pros

- quadratic convergence condition *without* knowing roots,
- approximates how far away you are

\[ |x - \xi| < 2\beta(f, x), \]
Pros

- quadratic convergence condition *without* knowing roots,
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\[ |x - \xi| < 2\beta(f, x), \]

Cons
Pros
- quadratic convergence condition *without* knowing roots,
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  \[ |x - \xi| < 2\beta(f, x), \]

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Pros
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- doesn’t say which root (but $\beta$ gives us an idea)
- alpha region much smaller than quad. conv. region
Ensure two \textit{approximate solutions} $x_1, x_2$ have distinct \textit{associated solutions} $\xi_1, \xi_2$. 
Question #2: Converging to Distinct Roots

Ensure two approximate solutions \( x_1, x_2 \) have distinct associated solutions \( \xi_1, \xi_2 \).

Smale Theorem #2

If

\[
|x_1 - x_2| > 2\left(\beta(f, x_1) + \beta(f, x_2)\right)
\]

then

\( \xi_1 \neq \xi_2 \)
Question #2: Converging to Distinct Roots

Ensure two approximate solutions $x_1, x_2$ have distinct associated solutions $\xi_1, \xi_2$.

Smale Theorem #2

If

$$|x_1 - x_2| > 2\left(\beta(f, x_1) + \beta(f, x_2)\right)$$

then

$$\xi_1 \neq \xi_2$$

Follows from Smale Theorem #1:

$$|x - \xi| \leq 2\beta(f, x)$$
Question #2: Converging to Distinct Roots

Ensure two approximate solutions \( x_1, x_2 \) have distinct associated solutions \( \xi_1, \xi_2 \).

**Smale Theorem #2**

If

\[
|x_1 - x_2| > 2 \left( \beta(f, x_1) + \beta(f, x_2) \right)
\]

then

\[
\xi_1 \neq \xi_2
\]

- Follows from Smale Theorem #1:

\[
|x - \xi| \leq 2\beta(f, x)
\]

- Homework: prove this
Let $f(x, y) = y^3 - x$. 
Application: Analytic Continuation

Let $f(x, y) = y^3 - x$.

- function of $y$ with $x$ as a parameter,
Let $f(x, y) = y^3 - x$.

- function of $y$ with $x$ as a parameter,
- given an $x$ we can find roots $y_1, y_2, y_3$ to $f(x, y) = 0$, 
Let $f(x, y) = y^3 - x$.

- function of $y$ with $x$ as a parameter,
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- **fact**: polynomial roots vary continuously as function of coefficients
Let \( f(x, y) = y^3 - x \).

- function of \( y \) with \( x \) as a parameter,
- given an \( x \) we can find roots \( y_1, y_2, y_3 \) to \( f(x, y) = 0 \),
- **fact**: polynomial roots vary continuously as function of coefficients

roots “above” \( x \): \( y_1(x), y_2(x), y_3(x) \)
Example: \( f(x, y) = y^3 - x \)

Let \( x_i \) range from \( x_0 = 1 \) to \( x_N = 8 \):

\[
\begin{align*}
y_1(1) &= 1 \\
y_2(1) &= e^{2\pi i/3} \\
y_3(1) &= e^{4\pi i/3} \\
&\vdots \\
y_1(8) &= 2 \\
y_2(8) &= 2e^{2\pi i/3} \\
y_3(8) &= 2e^{4\pi i/3}
\end{align*}
\]
Example: \( f(x, y) = y^3 - x \)

Let \( x_i \) range along the complex circle

\[
x(t) = e^{2\pi it} - 2 \\
t \in [0, 1]
\]
Computing These $y$-Paths

Let $y_1^{(i)}, y_2^{(i)}, y_3^{(i)}$ be the $y$-roots computed above $x_i$:

$$f(x_i, y) = 0.$$
Computing These $y$-Paths

Let $y_1^{(i)}, y_2^{(i)}, y_3^{(i)}$ be the $y$-roots computed above $x_i$:

$$f(x_i, y) = 0.$$ 

**Goal:** compute corresponding $y$-roots

$$y_1^{(i+1)}, y_2^{(i+1)}, y_3^{(i+1)}$$

above $x_{i+1}$: *solution to* $f(x_{i+1}, y) = 0$. 
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- **Idea:** use $y_1^{(i)}$ as Newton iteration guess in

  $$g(y) := f(x_{i+1}, y) = 0$$

  to get $y_1^{(i+1)}$.
Computing These $y$-Paths

Let $y_1^{(i)}, y_2^{(i)}, y_3^{(i)}$ be the $y$-roots computed above $x_i$:

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$$y_1^{(i+1)}, y_2^{(i+1)}, y_3^{(i+1)}$$

above $x_{i+1}$: solution to $f(x_{i+1}, y) = 0$.

▶ **Idea:** use $y_1^{(i)}$ as Newton iteration guess in

$$g(y) := f(x_{i+1}, y) = 0$$

to get $y_1^{(i+1)}$

▶ **Important:** must satisfy

$$y_1(x_i) = y_1^{(i)} \quad \text{and} \quad y_1(x_{i+1}) = y_1^{(i+1)}$$
Example: \( f(x, y) = y^3 - 2x^3y + x^7 \)

Let \( x_i \) range along the complex circle

\[ x(t) = e^{2\pi it} - 2 \]
\[ t \in [0, 1] \]

64 different \( x \)-values

small \( \Delta x \) means \( y^{(i)} \) are good guesses for \( y^{(i+1)} \)
Example: $f(x, y) = y^3 - 2x^3y + x^7$

Let $x_i$ range along the complex circle

$$x(t) = e^{2\pi it} - 2$$

$t \in [0, 1]$

16 different $x$-values

Something wrong happened. (Too large $\Delta x$.)
Problem

“Just take the $\Delta x$ steps to be really small.”
Problem

“Just take the $\Delta x$ steps to be really small.”

- What does “small” mean?
Problem

“Just take the Δx steps to be really small.”

- What does “small” mean?
- Heuristics in programming should be avoided. (Understatement of the year.)
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“Just take the $\Delta x$ steps to be really small.”

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- **Smale:** determine when $\Delta x$ is small enough such that
“Just take the $\Delta x$ steps to be really small.”

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- Heuristics in programming should be avoided. (Understatement of the year.)
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- Smale: determine when $\Delta x$ is small enough such that
  - each $y_j^{(i)}$ will converge under Newton
    (Use Smale Theorem #1)
“Just take the $\Delta x$ steps to be really small.”

- What does “small” mean?
- Heuristics in programming should be avoided. (Understatement of the year.)
- Too small $\rightarrow$ computationally inefficient.
- **Smale:** determine when $\Delta x$ is small enough such that
  - each $y_j^{(i)}$ will converge under Newton  
    (Use Smale Theorem #1)
  - each $y_j^{(i)}$ will converge to distinct $y_j^{(i+1)}$  
    (Use Smale Theorem #2)
Algorithm: Analytic Continuation

**Algorithm:** $\text{analytic}(f, x_i, x_{i+1}, y^{(i)})$
Algorithm: Analytic Continuation

Algorithm: analytic\((f, x_i, x_{i+1}, y^{(i)})\)

Input:

- polynomial \( f = f(x, y) \),
- \( x \)-points \( x_i \) and \( x_{i+1} \),
- ordered \( y \)-roots \( y^{(i)} = (y_1^{(i)}, \ldots, y_d^{(i)}) \) above \( x_i \).
Algorithm: Analytic Continuation

Algorithm: `analytic(f, x_i, x_{i+1}, y^{(i)})`

Input:
- polynomial $f = f(x, y)$,
- $x$-points $x_i$ and $x_{i+1}$,
- ordered $y$-roots $y^{(i)} = (y_1^{(i)}, \ldots, y_d^{(i)})$ above $x_i$.

Output: ordered $y$-roots $y^{i+1} = (y_1^{(i+1)}, \ldots, y_d^{(i+1)})$ above $x_{i+1}$.
- such that $y_j^{(i)} \rightarrow y_j^{(i+1)}$ (same position $j$)
Algorithm: Analytic Continuation

Algorithm: \texttt{analytic}(f, x_i, x_{i+1}, y^{(i)})

1. Check that each $y_j^{(i)}$ is an \textbf{approximate solution} to

   \[ g(y) := f(x_{i+1}, y) = 0 \]

   using $\alpha(g, y_j^{(i)}) < 0.157 \ldots$ If any are not, \textbf{refine step}: 

   \[ x_i^{(i+1)} \leftarrow \frac{x_i + x_{i+1}}{2} \]

   \[ y_j^{(i+1)} \leftarrow \texttt{analytic}(f, x_i^{(i+1)}, x_{i+1}, y_j^{(i+1/2)}) \]

   \[ y_j^{(i+1)} \leftarrow \texttt{analytic}(f, x_{i+1}, x_{i+1}, y_j^{(i+1/2)}) \]

2. Determine if all approximate solutions $y_j^{(i)}$ will converge to \textbf{distinct} associated solutions $y_j^{(i+1)}$:

   \[ |y_j^{(i)} - y_k^{(i)}| > 2^{(\beta(f, y_j^{(i)})) + \beta(f, y_k^{(i)}))}, \forall j, k = 1, \ldots, d. \]

   If any are not, \textbf{refine step}.

3. Finally, Newton iterate each $y_j^{(i)}$ to $y_j^{(i+1)}$ and return.
Algorithm: Analytic Continuation

Algorithm: \texttt{analytic}(f, x_i, x_{i+1}, y^{(i)})

1. Check that each $y_j^{(i)}$ is an \textbf{approximate solution} to

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Algorithm: Analytic Continuation

Algorithm: analytic\((f, x_i, x_{i+1}, y^{(i)})\)

1. Check that each \(y_j^{(i)}\) is an approximate solution to
   \[ g(y) := f(x_{i+1}, y) = 0 \]
   using \(\alpha(g, y_j^{(i)}) < 0.157\ldots\) If any are not, refine step:
   - \(x_{i+1/2} \leftarrow (x_i + x_{i+1})/2\)
   - \(y^{(i+1/2)} \leftarrow \text{analytic}(f, x_i, x_{i+1/2}, y^{(i)})\)
Algorithm: Analytic Continuation

Algorithm: analytic\((f, x_i, x_{i+1}, y^{(i)})\)

1. Check that each \(y_j^{(i)}\) is an approximate solution to
   \[ g(y) := f(x_{i+1}, y) = 0 \]
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   \(\begin{align*}
   & x_{i+1/2} \leftarrow (x_i + x_{i+1})/2 \\
   & y^{(i+1/2)} \leftarrow \text{analytic}(f, x_i, x_{i+1/2}, y^{(i)}) \\
   & y^{(i+1)} \leftarrow \text{analytic}(f, x_{i+1/2}, x_{i+1}, y^{(i+1/2)})
   \end{align*}\)

2. Determine if all approximate solutions \(y_j^{(i)}\) will converge to distinct associated solutions \(y_j^{(i+1)}\):
   \[ |y_j^{(i)} - y_k^{(i)}| > 2(\beta(f, y_j^{(i)}) + \beta(f, y_k^{(i)})), \forall j, k = 1, \ldots, d.\]
   If any are not, refine step.

3. Finally, Newton iterate each \(y_j^{(i)}\) to \(y_j^{(i+1)}\) and return.
Algorithm: Analytic Continuation

Algorithm: \text{analytic}(f, x_i, x_{i+1}, y^{(i)})

1. Check that each \( y_j^{(i)} \) is an \textbf{approximate solution} to

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using \( \alpha(g, y_j^{(i)}) < 0.157 \ldots \) If any are not, \textbf{refine step}:

\begin{itemize}
  \item \( x_{i+1/2} \leftarrow (x_i + x_{i+1})/2 \)
  \item \( y^{(i+1/2)} \leftarrow \text{analytic}(f, x_i, x_{i+1/2}, y^{(i)}) \)
  \item \( y^{(i+1)} \leftarrow \text{analytic}(f, x_{i+1/2}, x_{i+1}, y^{(i+1/2)}) \)
\end{itemize}

2. Determine if all \textbf{approximate solutions} \( y_j^{(i)} \) will converge to \textbf{distinct associated solutions} \( y_j^{(i+1)} \):

\[
|y_j^{(i)} - y_k^{(i)}| > 2(\beta(f, y_j^{(i)}) + \beta(f, y_k^{(i)})), \quad \forall j, k = 1, \ldots, d.
\]

If any are not, \textbf{refine step}. 
Algorithm: Analytic Continuation

Algorithm: \( \text{analytic}(f, x_i, x_{i+1}, y^{(i)}) \)

1. Check that each \( y_j^{(i)} \) is an **approximate solution** to

\[
g(y) := f(x_{i+1}, y) = 0
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using \( \alpha(g, y_j^{(i)}) < 0.157 \ldots \) If any are not, **refine step:**

\[
\begin{align*}
&\quad x_{i+1/2} \leftarrow (x_i + x_{i+1})/2 \\
&\quad y^{(i+1/2)} \leftarrow \text{analytic}(f, x_i, x_{i+1/2}, y^{(i)}) \\
&\quad y^{(i+1)} \leftarrow \text{analytic}(f, x_{i+1/2}, x_{i+1}, y^{(i+1/2)})
\end{align*}
\]

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|y_j^{(i)} - y_k^{(i)}| > 2(\beta(f, y_j^{(i)}) + \beta(f, y_k^{(i)})), \quad \forall j, k = 1, \ldots, d.
\]

If any are not, **refine step**.

3. Finally, Newton iterate each \( y_j^{(i)} \) to \( y_j^{(i+1)} \) and return.
Example: $f(x, y) = y^3 - 2x^3y + x^7$

Let $x_i$ range along the complex circle

$$x(t) = e^{2\pi it} - 2$$

$t \in [0, 1]$

16 different $x$-values

Smale guarantees we converge to the correct roots.
Example: \( f(x, y) = y^3 - 2x^3y + x^7 \)

Let \( x \) range along the complex circle

\[
x(t) = \frac{1}{2} e^{2\pi it} + \beta
\]

\( t \in [0, 1] \)

where

\( \beta \approx -0.8369 - 0.6081j \).

(Branch point of curve.)
Final Remarks

- Works for square systems of polynomials $f : \mathbb{C}^n \to \mathbb{C}^n$.
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  - Derivative $f' \to$ Jacobian $Df$. 

Definition of $\gamma(f, x)$: "max $\to$ sup".

Some simpler bounds on $\gamma$: results in much smaller $\alpha$-region.
Final Remarks

- Works for square systems of polynomials $f : \mathbb{C}^n \to \mathbb{C}^n$.
  - Derivative $f' \rightarrow$ Jacobian $Df$.
- Even works for smooth functions $f : \mathbb{C}^n \to \mathbb{C}^n$. 

Definition of $\gamma(f, x)$: "max $\rightarrow$ sup".
- Some simpler bounds on $\gamma$: results in much smaller $\alpha$-region.
Final Remarks

▶ Works for square systems of polynomials $f : \mathbb{C}^n \to \mathbb{C}^n$.
  ▶ Derivative $f' \to$ Jacobian $Df$.
▶ Even works for smooth functions $f : \mathbb{C}^n \to \mathbb{C}^n$.
  ▶ Definition of $\gamma(f, x)$: “max $\to$ sup”.

Some simpler bounds on $\gamma$: results in much smaller $\alpha$-region.
Final Remarks

- Works for square systems of polynomials $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$.
  - Derivative $f' \rightarrow$ Jacobian $Df$.
- Even works for smooth functions $f : \mathbb{C}^n \rightarrow \mathbb{C}^n$.
  - Definition of $\gamma(f, x)$: “$\max \rightarrow \sup$”.
  - Some simpler bounds on $\gamma$: results in much smaller $\alpha$-region.
Thank you


References